Irredundant Normal Forms and Minimal Dependence Sets of a Boolean Function

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C. HALATSIS AND N. GAITANIS

Abstract—A new method is presented for determining all minimal dependence sets, irredundant normal forms, and irredundant normal forms of minimal dependence sets of a Boolean function $f$. The method reduces the above problems into those of determining all minimal positive dependence sets, irredundant positive normal forms, and irredundant positive normal forms of minimal positive dependence sets, respectively, of a Boolean function $f^*$ corresponding to $f$. For each problem a corresponding cover formula is developed such that the prime implicants of it are in one-to-one correspondence to all possible solutions.

Index Terms—Boolean functions, dependence sets, irredundant normal forms, minimization, positive functions.

I. INTRODUCTION

The present paper is concerned with the following three problems: 1) the finding of all minimal dependence sets of a Boolean function, 2) the finding of all irredundant normal forms of a Boolean function, and 3) the finding of all irredundant normal forms of a Boolean function with given minimal dependence set.

Given a Boolean function $f$ and its irredundant normal form $R$, the set $D$ of variables (where variable $x$ and its complement $\bar{x}$ are considered to be different), on which $R$ depends is said to be a minimal dependence set of $f$ if and only if there is not irredundant normal form $R'$ of $f$ depending on a proper subset of $D$. According to Wood [1] the dependence set of a normal form of a Boolean function $f$ can be considered to be a measure of the complexity of the corresponding logic circuit, the minimal dependence sets ensuring, in general, minimal complexity of the corresponding logic circuits. Furthermore, the knowledge of the minimal dependence sets of a function $f$ is very useful in cases where fan-in restrictions are posed upon the realization of the corresponding logic circuit. This notion of minimal dependence sets of Boolean functions is further explored in [2] where it is used for the minimization of the control store of microprogrammed computers. As to the authors' knowledge no published work exists on the first problem. The solution presented here yields so-called dependence function, the prime implicants of which are in one-to-one correspondence to the minimal dependence sets of the Boolean function $f$.

The second problem has been treated extensively in the past (see [3]-[12]), the usual approach involving, in general, two distinct phases: 1) finding all prime implicants and 2) searching for irredundant sums of prime implicants (covering problem). Compared to past work on this problem the presented solution, from a structural point of view, differs in that it does not phase the problem in two, and is alike to those [8]-[12] yielding a cover formula the prime implicants of which are in one-to-one correspondence to the irredundant normal forms. However, from a conceptual point of view, the method is entirely different from those given in the past since it transforms the initial problem into that of finding all irredundant positive normal forms. Actually, given a Boolean function $f$ all three problems are resolved into finding respectively minimal positive dependence sets, irredundant positive normal forms, and irredundant positive normal forms with minimal positive dependence sets of a Boolean function $f^*$ corresponding to $f$. From this point of view the method treats all three problems uniformly and is insensitive to which form the function to be simplified is given, or to whether the function is completely or not specified.

In Section II all minimal positive dependence sets of a Boolean function are determined. Section III deals with the finding of all irredundant positive normal forms and those with minimal dependence sets. Lastly, Section IV deals with the transformation of the initial problems to those solved in Sections II and III.

II. MINIMAL POSITIVE DEPENDENCE SETS

Let $f: \{0, 1\}^n \rightarrow \{0, 1\}$ be a Boolean function and let $R^1, R^2: \{0, 1\}^* \rightarrow \{0, 1\}$ be Boolean functions such that

$$R^1(x) = 1 \iff x \in f^{-1}(1)$$
$$R^2(x) = 1 \iff x \in f^{-1}(0).$$

A function $R: \{0, 1\}^* \rightarrow \{0, 1\}$ is said to be equivalent to $f$ iff

$$R \leq R^1 \leq R^2.$$  

Let $F^1, F^2: \{0, 1\}^* \rightarrow \{0, 1\}$ be Boolean functions such that

$$F^1(x) = 1 \iff x \geq y \in f^{-1}(1)$$
$$F^2(x) = 1 \iff x \leq y \in f^{-1}(0).$$

Lemma 1: Let $R: \{0, 1\}^* \rightarrow \{0, 1\}$ be a positive Boolean function. $R$ is equivalent to $f$ iff

$$F^1 \leq R \leq F^2.$$

Definition 1: Let $f: \{0, 1\}^* \rightarrow \{0, 1\}$ be a Boolean function and $i_1, i_2, \ldots, i_m \subseteq \{1, 2, \ldots, n\}$, $i_1, i_2, \ldots, i_m$ is called a positive dependence set of $f$ iff there exists an irredundant positive normal form $f$ of $f$ depending on the $x_{i_1}, x_{i_2}, \ldots, x_{i_m}$ variables.

A positive dependence set $i_1, i_2, \ldots, i_m$ of $f$ is said to be minimal if there is no positive dependence set of $f$ proper subset of $i_1, i_2, \ldots, i_m$.

Theorem 1: Let $f: \{0, 1\}^n \rightarrow \{0, 1\}$ be a Boolean function and $i_1, i_2, \ldots, i_m \subseteq \{1, 2, \ldots, n\}$, $i_1, i_2, \ldots, i_m$ is a positive dependence set of $f$ iff for every prime implicant $P$ of $f$ and every prime implicant $Q$ of $F^2$ there exists $i_j \in \{i_1, i_2, \ldots, i_m\}$ such that $x_{i_j} \geq P$ and $x_{i_j} \geq Q$.

Proof: Let $i_1, i_2, \ldots, i_m$ be a positive dependence set of $f$. Then by Lemma 1 and Definition 1 there exists positive Boolean function $R$ such that

$$F^1 \leq F^1[i_1, i_2, \ldots, x_{i_1}, i_1, \ldots, 1] \leq R \leq F^2[0, \ldots, 0, x_{i_1}, x_{i_2}, \ldots, x_{i_m}, 0, \ldots, 0] \leq F^2.$$  

Then for every prime implicant $P$ of $F^1$ there exists prime implicant $p$ of

$$F^2[0, \ldots, 0, x_{i_1}, x_{i_2}, \ldots, x_{i_m}, 0, \ldots, 0]$$

such that $P \leq p$. But for every prime implicant $p$ of

$$F^2[0, \ldots, 0, x_{i_1}, x_{i_2}, \ldots, x_{i_m}, 0, \ldots, 0]$$

and every prime implicant $q$ of $F^2[0, \ldots, 0, x_{i_1}, x_{i_2}, \ldots, x_{i_m}, 0, \ldots, 0]$ there exists $i_j \in \{i_1, \ldots, i_m\}$ such that $x_{i_j} \geq P$ and $x_{i_j} \geq Q$, where $Q$ is a prime implicant of $F^2$. Thus for every prime implicant $P$ of $F^1$ and every prime implicant $Q$ of $F^2$ there exists $i_j \in \{i_1, \ldots, i_m\}$ such that $x_{i_j} \geq P$ and $x_{i_j} \geq Q$. 

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Conversely, assume that for every prime implicant $P$ of $F^j$ and every prime implicant $Q$ of $F^j_0$ there exists $i_j \in \{i_1, \cdots, i_m\}$ such that $x_{i_j} \geq P$ and $x_{i_j} \geq Q$. Then

$$F^j(1, \cdots, 1, x_{i_1}, \cdots, x_{i_m}, 1, \cdots, 1) = F^j_0(0, \cdots, 0, x_{i_1}, \cdots, x_{i_m}, 0, \cdots, 0) = 0$$

for every $x_{i_1}, x_{i_2}, \cdots, x_{i_m} \in \{0, 1\}$. Hence, there exists positive Boolean function $R$ that satisfies relation (1) and thus, by Lemma 1 and Definition 1 $\{i_1, \cdots, i_m\}$ is a positive dependence set of $f$. Let $P$ and $Q$ be positive and negative product terms, respectively. Denote by $P \Delta Q$ the logical sum given by

$$P \Delta Q = \sum \{x_i | x_i \geq P \text{ and } x_i \geq Q\}. \quad (2)$$

Denote also by $H_f$ the Boolean function given by

$$H_f = \prod_{j=1}^{k} P_i \Delta Q_j \quad (3)$$

where $P_i$, $i = 1, 2, \cdots, m$ and $Q_j$, $j = 1, 2, \cdots, k$ are the prime implicants of the functions $F^j$ and $F^j_0$, respectively. Then the following corollary is a consequence of Theorem 1.

**Corollary 1:** Let $f: \{0, 1\}^n \rightarrow \{0, 1\}$ be a Boolean function and $\{i_1, i_2, \cdots, i_m\} \subseteq \{1, 2, \cdots, n\}$. $\{i_1, i_2, \cdots, i_m\}$ is a minimal positive dependence set of $f$ if $x_{i_1}, x_{i_2}, \cdots, x_{i_m}$ is a prime implicant of $H_f$.

**Example 1:** Let $f: \{0, 1\}^n \rightarrow \{0, 1\}$ be a Boolean function such that

$$R^1_f = x_1x_2x_3x_4x_5x_6 + x_1x_2x_3x_4x_5x_6 + x_1x_2x_3x_4x_5x_6 + x_1x_2x_3x_4x_5x_6$$

$$R^2_f = x_1x_2x_3x_4x_5x_6 + x_1x_2x_3x_4x_5x_6 + x_1x_2x_3x_4x_5x_6 + x_1x_2x_3x_4x_5x_6$$

$$= x_1x_2x_3x_4x_5x_6 + x_1x_2x_3x_4x_5x_6 + x_1x_2x_3x_4x_5x_6 + x_1x_2x_3x_4x_5x_6.$$

Then

$$F^j_1 = x_4x_5x_6 + x_3x_4x_6 + x_1x_2x_3$$

$$F^j_2 = x_1x_2x_3x_4x_5x_6 + x_1x_2x_3x_4x_5x_6 + x_1x_2x_3x_4x_5x_6 + x_1x_2x_3x_4x_5x_6 + x_1x_2x_3x_4x_5x_6.$$

Thus, the irredundant normal form of $H_f$ is

$$H_f = x_1x_2x_4(x_5 + x_6)(x_4 + x_5) = x_1x_2x_4x_6 + x_1x_2x_3x_4x_5$$

and hence, $\{1, 2, 4, 6\}$ and $\{1, 2, 3, 4, 5\}$ are the minimal positive dependence sets of $f$.

**III. IRREDUNDANT POSITIVE NORMAL FORMS**

Let $f: \{0, 1\}^n \rightarrow \{0, 1\}$ be a Boolean function. Let also $P_i$ and $\{Q_1, Q_2, \cdots, Q_k\}$ be respectively a prime implicant of $F^j$ and the set of prime implicants of $F^j_0$. Denote by $H_p$ the Boolean function given by

$$H_p = \prod_{j=1}^{k} P_i \Delta Q_j.$$ \hspace{1cm} (4)

**Lemma 2:** $q$ is a prime implicant of $H_p$, iff $q$ is a prime implicant of $F^j_0$ such that $P_i \leq q$.

**Proof:** Let $q$ be a prime implicant of $H_p$. Then $q \geq P_i$ and for every prime implicant $Q_j$ of $F^j_0$ there exists $x_{i_j} \geq q$ such that $x_{i_j} \geq Q_j$. This implies that $q \leq P_i$ and that there is not $q' < q$ such that $q' \leq P_i$.

Hence, $q$ is a prime implicant of $F^j_0$ such that $P_i \leq q$.

Conversely, let $q$ be a prime implicant of $F^j_0$ such that $P_i \leq q$. Then for every $x_{i_j} \geq P_i$ there exists at least one prime implicant $Q_j$ of $F^j_0$ such that $q \Delta Q_j = x_{i_j}$ and $x_{i_j} \leq P_i \Delta Q_j$. Hence, $q \leq H_p$, and there is not $q' < q$ such that $q' \leq H_p$, that is, $q$ is a prime implicant of $H_p$.

The above lemma establishes a useful criterion for the determination of irredundant positive normal forms of a Boolean function $f$ starting from the irredundant normal forms of its $F^j$ and $F^j_0$.

Thus, for every prime implicant $P_i$, $i = 1, 2, \cdots, m$ of $F^j$ the functions $H_p$, $i = 1, 2, \cdots, m$ given by (4) are determined and put into the irredundant normal form

$$H_p = \prod_{i=1}^{m} P_i \Delta Q_j = q_1 + q_1 + \cdots + q_m \quad i = 1, 2, \cdots, m.$$ \hspace{1cm} (5)

Next, the Boolean formula $R_f$ given below is formed:

$$R_f = \prod_{H_p = x_{i_1} + \cdots + x_{i_m}} \left( z_{i_1} + z_{i_2} + \cdots + z_{i_m} \right)$$\hspace{1cm} (6)

where $z_{i_j}$ are distinct literals corresponding to each distinct $q_{i_j}$ term. Then Corollary 2 below gives all irredundant positive normal forms of $f$. The proof of the corollary is easily derived from Lemma 2.

**Corollary 2:** A formula $R = q_1 + q_2 + \cdots + q_n$ is an irredundant positive normal form of $f$ iff $z_1, z_2, \cdots, z_n$ is a prime implicant of $R_f$.

**Example 2:** Let $f: \{0, 1\}^n \rightarrow \{0, 1\}$ be the Boolean function of Example 1. Then

$$H_{x_4x_5x_6} = (x_5 + x_6)(x_4 + x_5)(x_4 + x_5) = x_4x_5 + x_4x_6 + x_4x_6$$

$$H_{x_3x_4x_5} = (x_3 + x_4)(x_4 + x_4) = x_3x_4 + x_4x_4$$

$$H_{x_1x_2x_3} = x_1x_2$$

and

$$R_f = (z_1 + z_2 + z_3)(z_4 + z_5) = z_2z_3 + z_1z_4 + z_5 + z_1z_4$$

Thus the positive irredundant normal forms of $f$ are

$$R_1 = x_4x_6 + x_1x_2$$

$$R_2 = x_4x_5 + x_3x_4 + x_1x_2$$

$$R_3 = x_3x_6 + x_3x_4 + x_1x_2.$$
contrast, with the present method, by Lemma 2 and the use of the unate functions \( F_j^1 \) and \( F_j^2 \), one effectively computes only those positive prime implicants of \( R_j^1 \) that cover at least one positive prime implicant of \( R_j^2 \). In this case the computational effort is small since the functions \( F_j^1 \) and \( F_j^2 \) contain in general less terms than the \( R_j^1 \) and \( R_j^2 \) functions. The next example illustrates particularly these points.

**Example 3:** Consider the Boolean function \( f : \{0, 1\}^5 \rightarrow \{0, 1\} \) defined by

\[
R_j^1 = x_1 x_2 x_3 x_4 x_5 + x_1 x_2 x_3 x_4 x_5 + x_1 x_2 x_3 x_4 x_5 \\
+ x_1 x_2 x_3 x_4 x_5 + x_1 x_2 x_3 x_4 x_5 \\
R_j^2 = x_1 x_2 x_3 x_4 x_5 + x_1 x_2 x_3 x_4 x_5 + x_1 x_2 x_3 x_4 x_5 \\
+ x_1 x_2 x_3 x_4 x_5 + x_1 x_2 x_3 x_4 x_5 \\
+ x_1 x_2 x_3 x_4 x_5 + x_1 x_2 x_3 x_4 x_5 \\
+ x_1 x_2 x_3 x_4 x_5 + x_1 x_2 x_3 x_4 x_5
\]

Then

\[
F_j^1 = x_2 x_4 + x_2 x_3 x_5 + x_1 x_4 x_5 \\
F_j^2 = x_2 x_4 + x_2 x_3 x_5 + x_1 x_4 x_5
\]

and

\[
x_2 x_4 \quad x_4 x_5 = x_4 \\
x_1 x_4 x_5 \quad x_4 x_5 = x_4 \\
x_1 x_4 x_5 \quad x_2 x_3 x_5 = x_2 \\
x_2 x_3 x_5 \quad x_2 x_3 x_5 = x_2 \\
x_2 x_3 x_5 \quad x_1 x_2 x_3 = x_2 \\
x_2 x_3 x_5 \quad x_1 x_2 x_3 = x_2
\]

Thus,

\[
H_{x_2 x_4} = x_2 x_4 = 1 \\
H_{x_2 x_3} = x_2 x_3 = 1 \\
H_{x_2} = x_2 = 1
\]

and

\[
R_f = z_1 z_2 z_3 z_4 = z_1 z_2 z_4 = z_1 z_2 z_3 z_4.
\]

Thus the positive irredundant normal forms of \( f \) are

\[
R_1 = x_2 x_4 + x_2 x_5 + x_1 x_5 \\
R_2 = x_2 x_4 + x_3 x_5 + x_1 x_5.
\]

The prime implicants of \( F_j^2 \) are \( x_1 x_5, x_2 x_4, x_3 x_4, x_3 x_5, x_3 x_4, x_3 x_5 \). Of these, the \( x_1 x_5 \) is not computed by our method.

Corollaries 1 and 2 may be combined to yield the irredundant positive normal forms of \( f \) with minimal positive dependence sets. Thus, let \( \{i_1, i_2, \ldots, i_n\} \) be a minimal positive dependence set of \( f \) derived according to Corollary 1. Then for every prime implicant \( P_i, \ i = 1, 2, \ldots, m \), of \( F_j^1 \{1, 2, \ldots, m, x_{i_1}, \ldots, x_{i_n}, 1, \ldots, 1\} \) functions \( H_j \) analogous to (5) may be determined using the prime implicants \( Q_j, \ j = 1, 2, \ldots, k \) of \( F_j^2 \{0, \ldots, 0, x_{i_1}, \ldots, x_{i_n}, 0, \ldots, 0\} \). The corresponding \( R_j \) function determines then, by Corollary 2, all irredundant positive normal forms \( R \) of \( f \) with \( \{i_1, i_2, \ldots, i_n\} \) minimal positive dependence set.

**Example 4:** Consider again the function \( f \) of Example 1. The set \( \{1, 2, 4, 6\} \) was found to be a minimal positive dependence set of \( f \). Then

\[
F_j^1(x_1, x_2, 1, x_4, 1, x_2) = x_4 x_6 + x_1 x_2 \\
F_j^2(x_1, x_2, 0, x_4, 0, x_2) = x_1 x_6 + x_1 x_4 + x_2 x_6 + x_2 x_4
\]

and

\[
H_{x_4 x_6} = x_4 x_6, \quad H_{x_1 x_2} = x_1 x_2 \\
\]

Thus, \( R_f = z_1 z_2 \) and hence \( R = x_4 x_6 + x_1 x_2 \) is the only irredundant positive normal form of \( f \) with \( \{1, 2, 4, 6\} \) minimal positive dependence set.

**IV. IRREDUNDANT NORMAL FORMS**

Let now, \( D \) be a one-to-one mapping from \( \{0, 1\}^n \) into \( \{0, 1\}^{2n} \) such that

\[
D : (x_1, x_2, \ldots, x_n) \rightarrow (x_1, x_2, \ldots, x_n, x_1, x_2, \ldots, x_n)
\]

(7)

Given a Boolean function \( f : \{0, 1\}^n \rightarrow \{0, 1\} \) let \( f^* : \{0, 1\}^{2n} \rightarrow \{0, 1\} \) be a Boolean function such that

\[
f^*(1) = D(f^{-1}(1)) \quad \text{and} \quad f^*(0) = D(f^{-1}(0)).
\]

The following propositions are obvious.

**Proposition 1:** Let \( R^* = R^*(x_{i_1}, x_{i_2}, \ldots, x_{i_n}) \) be a positive normal form of \( f^* \) and let \( R \) be the Boolean formula resulting from \( R^* \) by substituting every \( x_j \) with \( i_j > n \) by \( x_{i_j} \). Then \( R \) is a normal form of \( f \).

**Proposition 2:** Let \( R = R(x_{i_1}, \ldots, x_{i_n}) \) be a normal form of \( f \) and let \( R^* \) be the Boolean formula resulting from \( R \) by substituting every \( x_j \) by \( x_{i_j} \). Then \( R^* \) is a positive normal form of \( f \).

From the above it is deduced that the problem of finding all irredundant normal forms of a Boolean function \( f : \{0, 1\}^n \rightarrow \{0, 1\} \) reduces into that of finding all irredundant positive normal forms of the corresponding function \( f^* : \{0, 1\}^{2n} \rightarrow \{0, 1\} \) using the function \( R^* \).

Again as in the case of finding all irredundant positive normal forms of \( f \), the finding of all irredundant normal forms of \( f \) involves effectively, by Lemma 2, the computation among the prime implicants of \( R_j^2 \), which are the classical candidates for the covering problem only those prime implicants that cover at least one min-term of \( R_j^1 \). Example 6 below illustrates particularly this point.

**Definition 2:** Let \( f : \{0, 1\}^n \rightarrow \{0, 1\} \) be a Boolean function. A set \( \{i_1, i_2, \ldots, i_m\} \subseteq \{1, 2, \ldots, 2n\} \) is said to be a dependence set of \( f \) if an irredundant normal form of \( f \) depends on the \( x_{i_1}, x_{i_2}, \ldots, x_{i_m} \) variables (where \( x_{i_j} = x_{i_j} \) for \( i_j > n \)). A dependence set \( \{i_1, i_2, \ldots, i_m\} \) is said to be minimal if there is not irredundant normal form of \( f \) with dependence set proper subset of \( \{i_1, i_2, \ldots, i_m\} \).

From Propositions 1 and 2 it is obvious that a set \( \{i_1, i_2, \ldots, i_m\} \) is a dependence set of \( f \) iff \( \{i_1, i_2, \ldots, i_m\} \) is a positive dependence set of the function \( f^* \). A dependence set \( \{i_1, i_2, \ldots, i_m\} \) is a
### TABLE 1

<table>
<thead>
<tr>
<th>Positive Irrelevant Normal Forms of $f^*$</th>
<th>Irrelevant Normal Forms of $f$</th>
<th>Positive Irrelevant Normal Forms of $f^*$ with Minimal Positive Dependence Set</th>
<th>Irrelevant Normal Forms of $f$ with Minimal Dependence Set</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R_f$, Term</td>
<td>Form</td>
<td>Dependence Set</td>
<td>Form</td>
</tr>
<tr>
<td>$x_2 x_8$</td>
<td>$x_4 x_6 + x_1 x_2$</td>
<td>$x_4 x_6 + x_1 x_2$</td>
<td>$[1, 2, 4, 6]$</td>
</tr>
<tr>
<td>$x_2 x_4$</td>
<td>$x_3 x_5 + x_4 x_6$</td>
<td>$x_3 x_5 + x_4 x_6$</td>
<td>$[4, 5, 6, 9]$</td>
</tr>
<tr>
<td>$x_2 x_4$</td>
<td>$x_4 x_6 + x_1 x_2$</td>
<td>$x_4 x_6 + x_1 x_2$</td>
<td>$[6, 9, 10]$</td>
</tr>
<tr>
<td>$x_2 x_4$</td>
<td>$x_4 x_6 + x_1 x_2$</td>
<td>$x_4 x_6 + x_1 x_2$</td>
<td>$[1, 2, 4, 6]$</td>
</tr>
<tr>
<td>$x_2 x_4$</td>
<td>$x_3 x_5 + x_4 x_6$</td>
<td>$x_3 x_5 + x_4 x_6$</td>
<td>$[7, 8, 9, 10]$</td>
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<tr>
<td>$x_2 x_4$</td>
<td>$x_4 x_6 + x_1 x_2$</td>
<td>$x_4 x_6 + x_1 x_2$</td>
<td>$[3, 4, 5, 9]$</td>
</tr>
<tr>
<td>$x_3 x_5 + x_4 x_6$</td>
<td>$x_3 x_5 + x_4 x_6$</td>
<td>$x_3 x_5 + x_4 x_6$</td>
<td>$[1, 2, 3, 4]$</td>
</tr>
</tbody>
</table>

$i_2, \ldots, i_n$ of $f$ is minimal iff it is a minimal positive dependence set of $f^*$. Hence, the problem of finding all minimal dependence sets of $f$ reduces into that of finding all minimal positive dependence sets of $f^*$ using the function $H_{f^*}$, called henceforth dependence function of $f$.

Lastly, the problem of finding all irrelevant normal forms of $f$ with minimal dependence sets is reduced into that of finding all irrelevant positive normal forms of $f^*$ with minimal positive dependence sets.

**Example 5:** Consider again the function $f$ of Example 1. Using the mapping $D$ of (7) we have

$F^* = x_4 x_5 x_6 x_7 x_8 x_9 + x_3 x_4 x_6 x_7 x_8 x_9 x_{11} + x_1 x_2 x_3 x_9 x_{10} x_{12}$

$\equiv P_1 + P_2 + P_3$

$F_f = x_1 x_3 x_5 x_6 x_8 x_{10} + x_1 x_2 x_3 x_5 x_6 x_8 x_{11} + x_1 x_2 x_3 x_5 x_6 x_8 x_{10} x_{11}$

Then

$H_{f_1} = (x_5 + x_6 + x_8)(x_4 x_5 + x_6 + x_9)(x_4 x_5 + x_6 + x_9)$

Then

$H_{f_2} = (x_5 + x_6 + x_8)(x_4 x_5 + x_6 + x_9)$

and

$R_{f^*} = (z_1 + z_2 + z_3 + z_4 + z_5 + z_6 + z_7 + z_8 + z_9)(z_1 + z_2 + z_3 + z_4 + z_5 + z_6 + z_7 + z_8 + z_9)$

In Table 1 are listed all irrelevant positive normal forms of $f^*$ deduced from $R_f$, and all irrelevant positive normal forms of $f^*$ with minimal positive dependence sets determined from $H_{f^*}$. The irrelevant normal forms of $f$ and those with minimal dependence set, listed also in Table 1 are derived from the respective forms of $f^*$ by substituting $x_7, x_8, x_9, x_{10}, x_{11},$ and $x_{12}$ with $x_1, x_2, x_3, x_4$, and $x_5$, respectively.

**Example 6:** Consider again the function $f$ of Example 3. Using mapping $D$ of (7) we have

$F^*_f = x_2 x_4 x_6 x_8 x_{10} + x_2 x_3 x_5 x_6 x_9 + x_2 x_3 x_4 x_5 x_6$

Then

$H_{f_1} = (x_4 x_6 + x_8)(x_4 x_6 + x_9)$

Then

$H_{f_2} = (x_4 x_6 + x_8)(x_4 + x_7)(x_4 + x_7)$

In a similar way we find

$H_{f_4} = x_1 x_5 = z_9$

$H_{f_1} = x_2 x_4 = z_1$. 

and

$R_{f^*} = (z_1 + z_2 + z_3 + z_4 + z_5 + z_6 + z_7 + z_8 + z_9)(z_1 + z_2 + z_3 + z_4 + z_5 + z_6 + z_7 + z_8 + z_9)$

$= z_4 z_8 + z_2 z_4 + z_4 z_9 + z_4 z_8 + z_2 z_6 + z_4 z_9 + z_2 z_7$

$= z_4 z_8 + z_2 z_4 + z_4 z_9 + z_4 z_8 + z_2 z_6 + z_4 z_9 + z_2 z_7$

$= z_4 z_8 + z_2 z_4 + z_4 z_9 + z_4 z_8 + z_2 z_6 + z_4 z_9 + z_2 z_7$

$= z_4 z_8 + z_2 z_4 + z_4 z_9 + z_4 z_8 + z_2 z_6 + z_4 z_9 + z_2 z_7$

$= z_4 z_8 + z_2 z_4 + z_4 z_9 + z_4 z_8 + z_2 z_6 + z_4 z_9 + z_2 z_7$
Then
\[ R_{f^*} = z_1 z_6 (z_2 + z_4 + z_5 + z_6 + z_7) \]
\[ = z_1 z_6 z_2 + z_1 z_6 z_4 + z_1 z_6 z_5 + z_1 z_6 z_6 + z_1 z_6 z_7. \]

Thus, the irredundant normal forms of \( f \) are

\[ R_1 = x_1 x_4 + x_2 x_4 + x_1 x_3 \]
\[ R_2 = x_1 x_4 + x_2 x_4 + x_1 x_3 \]
\[ R_3 = x_1 x_5 + x_2 x_4 + x_3 x_5 \]
\[ R_4 = x_1 x_5 + x_2 x_4 + x_3 x_5 \]
\[ R_5 = x_1 x_5 + x_2 x_4 + x_4 x_5. \]

Observe that the prime implicant \( x_3 x_5 \) of \( R_{f^*}^T \) (or equivalently \( x_3 x_5 \) of \( F_{f^*}^T \)) is not computed by our method since it does not cover any minterm of \( R_1 \) (or equivalently any prime implicant of \( F_{f^*}^T \)).

V. CONCLUSIONS

The present work shows how the problems of finding all minimal dependence sets, irredundant normal forms, and irredundant normal forms of minimal dependence sets of a Boolean function \( f \) are transformed into corresponding ones of a positive Boolean function \( f^* \). Given \( f \) the bounding functions \( F_{f^*} \) and \( F_{f^*}^T \) of \( f^* \) are easily determined due to the monotone character of them, especially when \( f = \langle g, h \rangle \) is an incompletely specified function. The formula manipulations required by the method are also easy to carry out since they refer to positive formulas. The introduced operation \( A \) between a positive and a negative term [see (2)], quite easy by hand, resolves also into a complementation and an ANDing on a digital computer. Lastly, it would be of interest to consider extension of the method for the simplification of multi-output switching networks combinational ones or not.

REFERENCES


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On Multiple Operand Addition of Signed Binary Numbers

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Abstract—Recent application of negabinary number systems in signal processing has evoked the question of the suitability of binary base. Many proposals for multiple operand addition of unsigned binary numbers are available in the literature. Here, the addition of two numbers in 2's complement notation has been extended to \( n \) signed summands. The time delay remains the same as that of processing \( n \) unsigned numbers. This method shows new promises for its application to signal processing.

Index Terms—Multiadds, signal processing, signed numbers, 2's complement extension digits.

INTRODUCTION

Recently, the negabinary system has been advocated for signal processing [1]–[3] and other areas [4], [5] on the grounds that it has a sign-independent representation and hence is suitable for simultaneous addition of several numbers. Many papers [6]–[8] have dealt with multiple operand addition of unsigned binary numbers, but relatively little has appeared on the addition of signed binary numbers. Here we have extended the additive algorithm of numbers in 2's complement notation to multiple operand. Required modifications are also derived, and practical implementation steps are clearly indicated.

WORD-LENGTH EXTENSION REQUIREMENT

In the 2's complement number system, the most significant bit (MSB) always indicates the sign. There is no loss of generality to confine our discussion to integers. Let a number \( A \) be denoted by the \( k \)-tuple \( A \) as

\[ A \mapsto (a_{k-1}a_{k-2} \cdots a_0). \quad (1) \]

Then

\[ A = \sum_{i=0}^{k-2} a_i2^i - a_{k-1}2^{k-1}. \quad (2) \]

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